ENTROPY-LIKE FUNCTIONS IN MULTI-SCALE HIERARCHICAL ORGANIZATION

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EXTENDED ABSTRACT

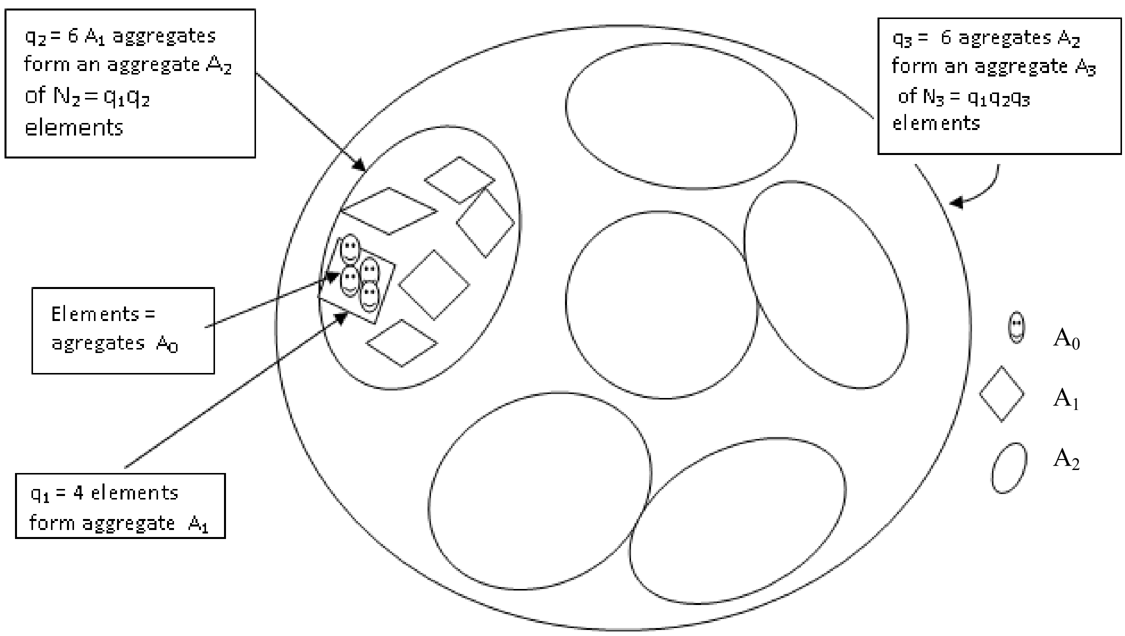
Multi-scale hierarchical organization of a population is understood here as the grouping of elements of this population into aggregates, these aggregates being grouped themselves into higher order aggregates, and so forth, to form a multi-scale organization. The characterization of such systems has been considered for various situations, going from particle clustering in fluidized beds, to collective behavior of micro-organisms in ecology, and including some attempts at modeling human organizations.

Entropy seems an appropriate concept to characterize order or organization. However, the classical statistical approach is not immediately transposable to this situation. The subject of the present communication is to illustrate the nature of the difficulty, which somehow connects to the non-extensivity issue addressed by the "new" entropies of Tsallis or Renyi.

The situation investigated differs from that of classical statistical thermodynamics in several respects. First, the multi-scale character requires a specific combinatorial analysis of particular ensembles, leading to a reformulation of the classical statistical entropy. Second, we are not dealing here with organization along dimensional physical quantities such as time, space or energy, but with organization in numbers. The corresponding entropy is therefore somehow analogous to probabilistic entropy or to informational entropy. Third, we are looking for the most ordered configurations, thus for distributions that minimize total entropy, rather than maximizing it. Fourth, we are considering systems with a small number of elements, in which the classical large number limits are not applicable or not useful.

Defining an entropy-like function that has suitable properties seems rather arbitrary. The approach explored here consists in defining entropies of the different scales in a nearly conventional way, and then weighting their contributions to account for their interdependence. Of course, this approach is not unique.

As an illustration, the value of trial functions having minimal properties is calculated with a very simple example, concerning the homogeneous organization of Ntot =144 "elements" in n = 3 levels of aggregation such that Ntot = N3= q1q2q3 , where q1 is the number of elements in the first level aggregates A1, q2 is the number of aggregates A1 in the second level aggregates A2 and q3 is the number of A2 aggregates in the largest aggregate A3, as illustrated in the figure.



The choice of 144 is because this relatively small number has many integer divisors, leading to 90 triplet configurations of aggregates with integer numbers of elements or sub-aggregates. In the figure, we have the triplets **q** = (4,6,6), ***N*** = (4, 24, 144)

In this example, the entropy of an aggregation level is defined from the number of permutations of the elements within an aggregate of this level, multiplied by the number of aggregates. The total entropy is then assumed to be a weighted sum of the entropies of each aggregation level, so that we have for example:

$$S_{tot} = \sum_{k=1}^{n} S_k = \sum_{k=1}^{n} \lambda_k q_{k+1\dots} q_n Ln[q_k!] = \sum_{k=1}^{n} \lambda_k \frac{N}{N_k} Ln\left[\frac{N_k}{N_{k-1}}!\right]$$

with: n=3; N_0=1; N_n= N_{tot}, while λ_k is a weighting factor discussed below.

The total entropy S_{tot} can now be computed for the 90 configurations, assuming values of the weighting coefficients, and one may look for the configuration(s) that gives the smallest entropy. For example, taking first λ_k =1, we find that the following triplets have total entropies distinctly smaller than all others, and in addition the different terms of the sum have similar orders of magnitude.

$\mathbf{q_a}$ = (2,4,18); S_a= 143.5 (18 A_2 aggregates, each containing 4 A_1 aggregates, each containing 2 elements)

$\mathbf{q_b}$ = (3,4,12); S_b= 144.1

Note that taking λ_i = 1 amounts to adding up the entropies of the different levels as if they were independent, thus to ignore the embedding of these levels. Taking different values for the λ is a non-unique way to account for the hierarchical structure, and results in considerable changes in the "minimal" configuration. For example, taking the λ triplet as $\boldsymbol{\lambda}$ = (1,4,8), one finds the following minimal distribution: **q** = (9,4,4) i.e. 4 A_2 aggregates, each containing 4 A_1 aggregates, of 9 elements each). This is a much more pyramidal configuration corresponding more to a human organization such as a research laboratory.

Other trial functions are explored: for example, a non-logarithmic function (possibly related to Tsallis' entropy) based on the number of **couples** of sub-aggregates in an aggregate ½ q(1-q). It is found that with the same weighting factors, the minimal entropy configurations are almost the same as the above.