THERMODYNAMICS FROM THE PERSPECTIVE OF INFORMATION GEOMETRY

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EXTENDED ABSTRACT

The statistical foundation of equilibrium thermodynamics relies heavily on the Gibbs distribution and its dependence on a few intensive variables such as the inverse temperature β and the chemical potential μ. In mathematical statistics these probability distributions are said to belong to the exponential family of models. In systems out of equilibrium the rationale for Gibbs distributions disappears. Other types of modeling, such as those tried out in the area of complex systems, are indicated.

My approach is based on generalizing the concept of an exponential family [1]. In the simplest case this is the q-exponential family as found in non-extensive statistical physics [2]. Here q is a deformation parameter. It equals 1 in the standard case. The same family is known in the domain of information geometry as Amari's alpha-family [3]. The concept has been further generalized in [4; 5] and applies in the context of both classical and quantum systems.

An essential characteristic of these generalized theories is that the dual structure of thermodynamics survives. The model states satisfy the maximum entropy principle. The dual of the inverse temperature β is the energy U. A Massieu function $\Phi(\beta)$ replaces the logarithm of the partition sum. Its contact transform is the thermodynamic entropy $S(U)$.

The main tool of the geometric formalism is a divergence function, which is the relative entropy functional of the physics literature. It is used as a distance measure. Arbitrary states are projected onto the manifold of model states by minimizing the divergence. The projection can be claimed to be orthogonal because a Pythagorean relation holds.

A nice example of the geometric approach is found in a study of the porous medium equation [6]. There it is proved that the projection of an arbitrary solution onto the statistical manifold of q-Gaussian distributions follows a geodesic. This suggests that it suffices to study the dynamics of the model states to capture the essence of non-equilibrium thermodynamics.

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